New Bounds for the Flock-of-Birds Problem

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The problem

The flock-of-birds problems

- A flock of n birds. Are there at least t birds?
- Each with a sensor. A sensor has capacity of Q states.
- from time to time, two birds come close;
- their sensors update their states according to a transition function $\delta \colon Q \times Q \to Q \times Q$.
- *(Population protocol)* You program δ! And a partition of the set of states into 0-states and 1-states such that...
- In any "realistic" infinite sequence of encounters
  - if $n < t$, eventually all sensors are forever in 0-states;
  - if $n \geq t$, eventually all sensors are forever in 1-states.

It should for work for any n with fixed Q! What is the minimal Q for a given threshold t?

Example of a population protocol

Before defining “realistic” sequences...

- Initially, all sensors are in 0-states and have 1 coin.
- When two sensors meet, one of them gets all the coins of the other one...
- unless they have $\geq t$ coins in total.
- In this case, they transit into a unique 1-state.
- sensors in the 1-state convert other sensors.
- $t+1$ states: $0, 1, \ldots, t-1$ coins (0-states) and the 1-state.
- $Q = t + 1$.

If $n < t$, all sensors will always be in 0-states.

If $n \geq t \ldots$

Realistic sequences

Definition

An infinite sequence of encounters $\alpha \in (\{1, 2, \ldots, n\}^2)^\omega$ is **realistic** if all words from $(\{1, 2, \ldots, n\}^2)^*$ has infinitely many occurrences in α.

Definition

A population protocol **solves the flock-of-birds problem with threshold** t if for every n:

- if $n \geq t$, then for all realistic infinite sequences of encounters of n birds eventually all sensors are always in 1-states;
- if $n < t$, then for all realistic infinite sequences of encounters of n birds eventually all sensors are always in 0-states.

Definition

A population protocol which solves the flock-of-birds problem with threshold t is **one-sided** if for every $n < t$, no sensor can ever come into a 1-state.

The problem

$Q(t)$ is the minimal Q such that there exists a population protocol with Q states solving the flock-of-birds problem with threshold t.

$Q_1(t)$ is the minimal Q such that there exists a *one-sided* population protocol with Q states solving the flock-of-birds problem with threshold t.

Context and Results

Population protocols

- Distributed computing, networks of mobile sensors, chemical reaction;
- LOGIC;
- Generally, population protocols are meant for computing *predicates* over natural numbers (not only unary).
- the flock of birds – threshold predicates $R(n) = \mathbb{I}\{n \geq t\}$.
- Theorem [Angluin et al., 2007]: a predicate can be computed by a population protocol $\iff$ this predicate is definable in Presburger arithmetic.
- idea: addition is easy.
- one-sided population protocols exactly compute threshold predicates and the all-0 predicate.

What next?

Minimizing:

the number of states

time of convergences

for a given predicate (and other kinds of problems like leader election).

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Minimization of the number of states

Theorem (Blondin et al., 2020)

Any predicate definable by a quantifier-free Presburger formula of bit-length l (assuming constants are written in binary) can be computed by a population protocol with $\mathrm{poly}(l)$ states.

$R(n) = \mathbb{I}\{n \geq t\}$ has bit-length $\log_2 t$
$\implies Q(t) \leq \mathrm{polylog}(t)$.

Results for the flock-of-birds

- $Q(t) = \Omega\left(\frac{\sqrt{\log(t)}}{\log\log(t)}\right)$ for infinitely many t by the counting argument.
- $Q(t) \geq \Omega(\log\log\log(t))$ for all t [Czerner and Esparza, 2021]
- $Q(t) \leq Q_1(t) \leq 4\log_2(t)$ for all t [Blondin, Esparza, Jaax, 2018]
- $Q_1(t) \geq \log_3(t)$ for all t [Blondin, Esparza, Jaax, 2018]
- $Q(t) = O(\log\log t)$ for infinitely many t [Cherner, 2022]

Theorem (Our results)

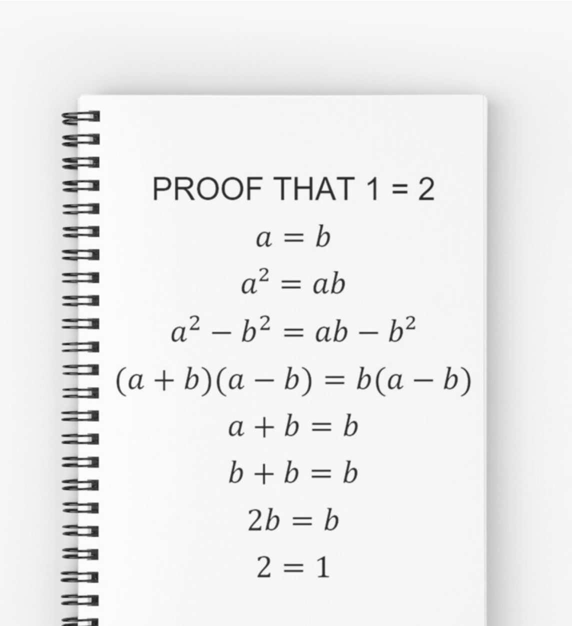
For all t we have

$$\log_2 t \leq Q_1(t) \leq \log_2 t + \min\{e, z\}$$

where e is the number of 1's in the binary expansion of t, and z is the number of 0's in the binary expansion of $t-1$.

Corollary: $\log_2 t \leq Q_1(t) \leq \frac{3}{2}\log_2 t$.

Overviews of the Proofs



More convenient definition

Definition

An infinite sequence of encounters $\alpha \in (\{1, 2, \ldots, n\}^2)^\omega$ is **realistic** if all words from $(\{1, 2, \ldots, n\}^2)^*$ has infinitely many occurrences in α.

Definition

A population protocol **solves the flock-of-birds problem with threshold** t if for every n:

- if $n \geq t$, then for all realistic infinite sequences of encounters of n birds eventually all sensors are always in 1-states;
- if $n < t$, then for all realistic infinite sequences of encounters of n birds eventually all sensors are always in 0-states.

Configuration graphs

A population protocol Π. **A configuration** is a vector in $\mathbb{N}^Q$ (how many sensors are in each state).

$C_1 \to C_2$ if some encounter brings C_1 to C_2. **Configuration graph** $G_n(\Pi)$.

Theorem

A population protocol Π solves the flock-of-birds problem with threshold $t \iff$ for every n, for every "trap" in $G_n(\Pi)$ the following holds. If $n \geq t$, then all configurations in this trap have only 1-states. And if $n < t$, then all configuration in this trap have only 0-states.

A trap – a reachable strongly connected component which is impossible to leave.

Lemma which implies the equivalence result

Lemma

For any realistic sequence of encounters, the set of configurations that occur infinitely often in it is a trap.

Theorem

A population protocol Π *solves the flock-of-birds problem with threshold* $t \iff$ *for every* n*, for every "trap" in* $G_n(\Pi)$ *the following holds. If* $n \geq t$*, then all configurations in this trap have only* 1*-states. And if* $n < t$*, then all configuration in this trap have only* 0*-states.*

Proof of the Lemma

Lemma

For any realistic sequence of encounters, the set of configurations that occur infinitely often in it is a trap.

All these configurations are reachable from one another. Only have to show that no other configuration is reachable.

The point: if you can leave a pairwise connected set of configurations S, then there exists a single finite sequence of encounters that leaves S from *any* configuration of S.

Example of an argument

- Initially, all sensors are in 0-states and have 1 coin.
- When two sensors meet, one of them gets all the coins of the other one...
- unless they have $\geq t$ coins in total.
- In this case, they transit into a unique 1-state.
- sensors in the 1-state convert other sensors.
- $t+1$ states: $0, 1, \ldots, t-1$ coins (0-states) and the 1-state.
- $Q = t + 1$.

If $n < t$, all sensors will always be in 0-states.

If $n \geq t \ldots$ Consider any trap. You can bring all into the 1-state. But you are still in the trap. So you can reach all configurations of the trap again. So there are only 1-states.

Upper bound

Theorem (Our results)

For all t we have

$$Q_1(t) \leq \log_2 t + \min\{e, z\}$$

where e is the number of 1's in the binary expansion of t, and z is the number of 0's in the binary expansion of $t-1$.

$$\begin{aligned} t &= 2^d \\ e &= 1 \\ z &= 0 \\ Q_1 &= d + O(1) \end{aligned} \qquad\qquad \begin{aligned} t &= 2^d - 1 \\ e &= d - 1 \\ z &= 1 \\ Q_1 &= d + O(1) \end{aligned}$$

Case $t = 2^d$

- the same protocol as before, but sensors can only hold *powers of two* of coins;
- If $n < 2^d$, the argument is the same.
- If $n \geq 2^d$, consider any trap.
- We should be able to bring some sensor into the 1-state. Then the argument is the same.
- Consider a configuration minimizing the number of non-bankrupt sensors.
- There has to be two equal powers of 2. Otherwise $1 + 2 + \ldots + 2^{d-1} < 2^d$.
- If it is $2^{d-1}, 2^{d-1}$, we can bring them into the 1-state.
- Otherwise, we could decrease the number of non-bankrupt sensors.

Case $t = 2^d - 1$

- the same protocol with *powers of two*;
- when two sensors with 2^{d-2} coins meet, one gets 2^{d-1} and the other one gets 1 coin **out of nowhere**
- If two sensors with 2^{d-1} coins meet, they both come into the 1-state.
- the case $n < t = 2^d - 1$;
- when a coin out of nowhere appears for the first time, we get one sensor with 2^{d-1} coins.
- $n - 2^{d-1} + 1 < 2^d - 1 - 2^{d-1} + 1 = 2^{d-1}$ coins in other sensors.
- can never have two sensors with 2^{d-2} coins again.

Case $t = 2^d - 1$

- case $n \geq t$.
- consider any trap. We have to bring somebody into the 1-state.
- maximize the total number of coins.
- then minimize the total number of non-bankrupt sensors.
- W.l.o.g. no two sensors with the same powers of 2.
- Maximally $1 + 2 + \ldots + 2^{d-1} = t$ coins.
- If we do not have 2^{d-1} coins, then we have less than t. If we have, then a coin out of nowhere was created at least one, so in the beginning we had less than t.

Lower bound

Theorem (Our results)

For all t we have

$$Q_1(t) \geq \log_2 t.$$

- Let S be the set of states of a one-sided population protocol Π, solving the FOB problem with threshold t.
- For $s \in S$, let $f(s)$ be the minimal n such that s can occur from the initial configuration of n birds.
- $|S| \geq |f(S)|$.
- $1 \in f(S)$ (because of the initial state);
- $t \in f(S)$ (because of some 1-state);
- $f(S)$ cannot have large gaps.

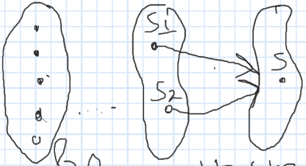
Lemma

If $a < b$ are two consecutive elements of $f(S)$, then $b \leq 2a$

Modulo Lemma, we must have about $\log_2(t)$ elements of $f(S)$ between 1 and t.

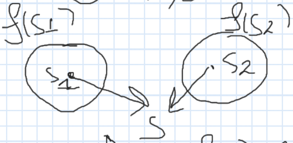
Lemma For any 2 consecutive elements $a < b$ of S, we have $b \le 2a$

Proof $\exists s$ s.t. $f(s) = b$



b least # steps

$f(s_1), f(s_2) \le a$



$b \le f(s) \le f(s_1) + f(s_2) = 2a$

Thank you!