State Complexity of Chromatic Memory in Infinite-Duration Games

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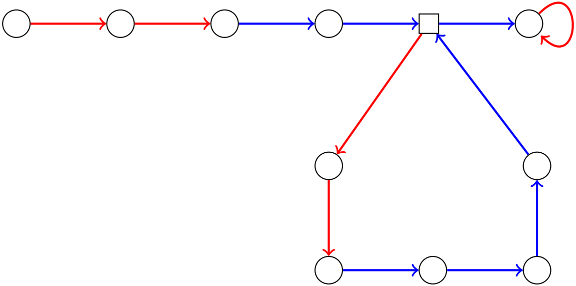
The problem

Example: axe



- Maximize the number of consecutive blue edges.
- How much memory? 2 states - whether you have already been to the square.
- What if you observe only colors? Then ≥ 3 states, because no 2-state DFA distinguishes RRBB and RRBBRRBBBB.

Chromatic vs. General Memory

- Edge-colored graphs.
- Some nodes might be controlled by an adversary (so we play a game).
- Infinite duration – so a play produces an infinite sequence of colors.
- Our goal – this infinite sequence has to be from our *winning condition* $W \subseteq C^\omega$.

Chromatic vs. General Memory

Non-uniform setting:

- **GenMem**(G, W) – the minimal number q s.t. there is a strategy with q states of general memory which wins w.r.t. W in the game graph G.
- **ChrMem**(G, W) – the same, but only chromatic strategies (can observe only colors of edges).
- **GenMem**$(G, W) \leq$ **ChrMem**(G, W), what else?

Uniform setting:

- **GenMem**$(W) = \max$ **GenMem**(G, W) over all G where we have some winning strategy w.r.t. W.
- **ChrMem**$(W) = \max$ **ChrMem**(G, W) over all G where we have some winning strategy w.r.t. W.
- **GenMem**$(W) \leq$ **ChrMem**(W), what else?

History, Motivation

Muller conditions

- A winning condition $W \subseteq C^\omega$ is **Muller** if the following holds: whether or not $c_1 c_2 c_3 \ldots \in C^\omega$ belongs to W is determined by the set of colors that occur infinitely often in $c_1 c_2 c_3 \ldots$.
- E.g., exactly two colors appear infinitely often.
- $\textbf{ChrMem}(W) < +\infty$ for every Mulller condition W (deterministic parity automata for Muller languages + positional determinacy of parity games).

Muller Conditions 2

Theorem (Casares, Colcombet, Lehtinen 2021-2022)

For any Muller condition W

- **ChrMem**(W) *equals the minimal size of a* **deterministic** *Rabin automaton, recognizing W.*
- **GenMem**(W) *equals the minimal size of a* **good-for-games** *Rabin automaton, recognizing W.*

Example of Casares:

$$\mathbf{ChrMem}(\text{exactly 2 colors}) = \#\text{ of colors},$$
$$\mathbf{GenMem}(\text{exactly 2 colors}) = 2.$$

Resolves a conjecture of Kopczyński(2006)

General Conditions

Theorem (Bouyer, Le Roux, Oualhadj, Randour, Randour, Vandenhove 2020)

For any winning condition W, if $\mathbf{ChrMem}(W) < +\infty, \mathbf{ChrMem}(\neg W) < +\infty$ *for graphs* **without** *adversary, then* $\mathbf{ChrMem}(W) < +\infty, \mathbf{ChrMem}(\neg W) < +\infty$ **with** *adversary.*

- Characterization of the class of W with $\mathbf{ChrMem}(W) < +\infty, \mathbf{ChrMem}(\neg W) < +\infty$.
- Open: $\mathbf{GenMem}(W) < +\infty \implies \mathbf{ChrMem}(W) < +\infty$? (for finitely many colors).

Results

Our results

Theorem (Upper bound)

For any W and G with n nodes, we have
$\mathbf{ChrMem}(G,W) \leq \left(\mathbf{GenMem}(G,W)+1\right)^n$.

Exponential improvement over [Le Roux, 2020]:

$$\mathbf{ChrMem}(G,W) \leq 2^{\mathbf{GenMem}(G,W)\cdot(n^2+1)}.$$

Theorem (Lower Bound)

For any n and q there exists W and G with n nodes such that

$$\mathbf{GenMem}(G,W) = q \text{ and } \mathbf{ChrMem}(G,W) \geq q^{n-3}.$$

Overview of the Proofs

Lower Bound

A **self-verifying automaton** is a NDFA with a partition of its states into *neutral*, *accepting* and *rejecting* states such that for any input word w **exactly one** of the following two statements hold:

- there exists a run of our automaton on w which leads to the accepting state;
- there exists a run of our automaton on w which leads to the rejecting state.

Theorem (Jirásková and Pighizzini, 2011)

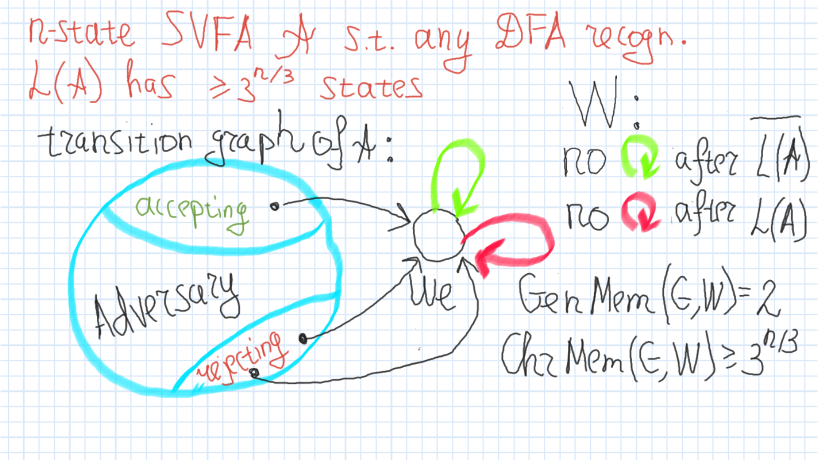
For any n there exists a language recognized by some n-state SVFA such that any DFA recognizing it has at least $3^{n/3}$ states.

Corollary (Weak Lower Bound)

For any n and q there exists W and G with $n+1$ node such that

$$\textbf{GenMem}(G, W) = 2 \text{ and } \textbf{ChrMem}(G, W) \geq 3^{n/3}.$$

Lower Bound


n-state SVFA A s.t. any DFA recogn.
L(A) has ≥ 3^{n/3} states
transition graph of A:
accepting
Adversary
rejecting
We
W:
no after $\overline{L(A)}$
no after L(A)
GenMem(∈,W) = 2
ChrMem(∈,W) ≥ 3^{n/3}

Upper bound

Theorem

For any W and G with n nodes, we have
$\mathbf{ChrMem}(G, W) \leq \big(\mathbf{GenMem}(G, W) + 1\big)^n$.

Plan:

A strategy S_1 with q states of general memory
$\mapsto$ A strategy S_2 with $(q+1)^n$ states of chromatic memory
s.t. $\mathrm{col}(S_2) \subseteq \mathrm{col}(S_1)$.

If S_1 is winning for W, then so is S_2, because

$$\mathrm{col}(S_2) \subseteq \mathrm{col}(S_1) \subseteq W.$$

Upper bound

- S_2 has to know "what would S_1 do in this situation".
- For a node u – a state of S_1 such that some play with S_1 comes to u and has the same sequence of colors.
- For some of the nodes we maintain such a state of S_1. For others "we don't know". So we need $(q+1)^n$ states.
- Our actual current node has to have a state.

Why is this sufficient? We have $\mathrm{col}(S_2) \subseteq \mathrm{col} S_1$ for finite paths, and hence for infinite by König's Lemma.

Upper bound

- S_2 has to know "what would S_1 do in this situation".
- For a node u – a state of S_1 such that some play with S_1 comes to u, has the same sequence of colors and brings memory S_1 into this state.
- For some of the nodes we maintain such a state of S_1. For others "we don't know". So we need $(q+1)^n$ states.
- Our actual current node has to have a state.

If we receive a color c, we look for all ways we can extend our current knowledge by a c-colored edge.

Thank you!